CHAPTER 1:

A SYSTEM OF COORDINATES FOR 3D

1.1 DEFINING AND USING COORDINATES

DEFINING A SYSTEM OF COORDINATES IN 3-D

If we are centered in San Juan, Puerto Rico, we can define a system of coordinates in the following manner:

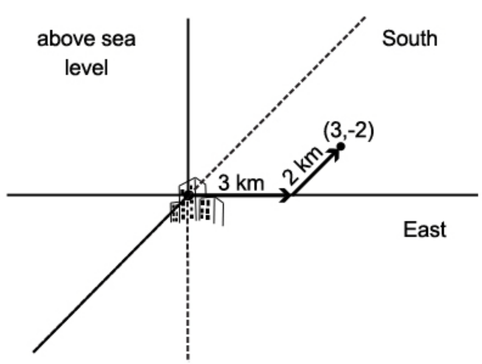
x = km west/east of San Juan: (-) is west, (+) is east.
y = km south/north of San Juan: (-) is south, (+) is north.
z = altitude in km: (-) is below sea level, (+) is above sea level.

This system will allow us to quickly associate a location with its corresponding x, y, and z values and vice versa. It is worth nothing that $(x, y, z) = (0, 0, 0)$ indicates that we are on the ground in San Juan.

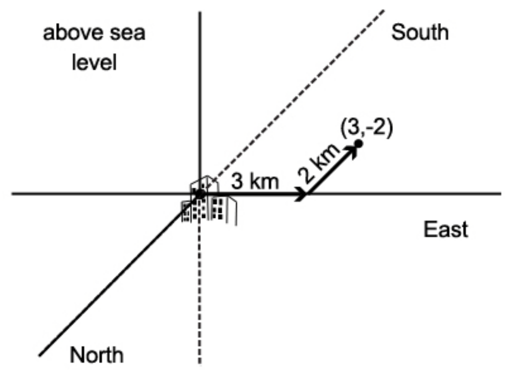
USING A SET OF COORDINATES TO IDENTIFY A LOCATION

Example Exercise 1.1.1: Find the location with coordinates $(x, y, z) = (3, -2, 4)$.

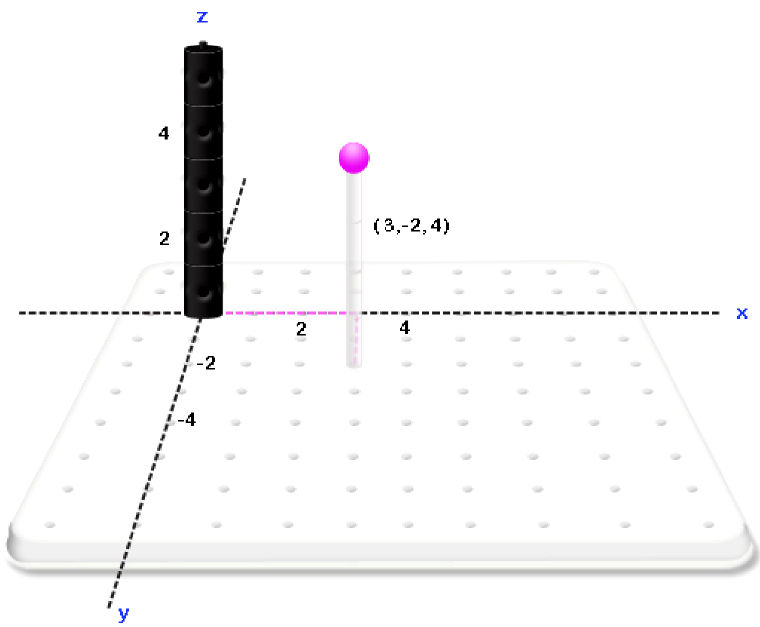
Solution: As $x = 3$, we can start in SJ and go 3 km east of San Juan.



Since $y = -2$, we must then go 2 km south of San Juan. This gives us the location in the xy plane associated with $(x, y) = (3, -2)$.



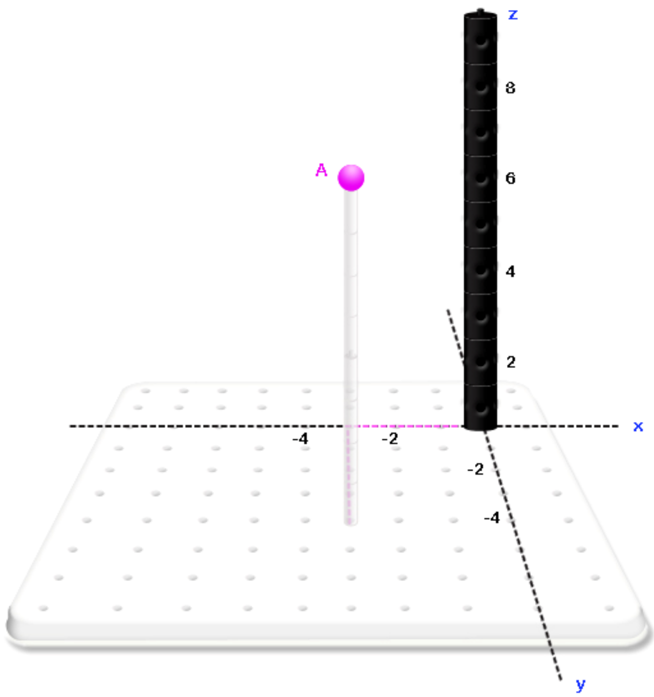
Finally, the new element of our system of coordinates is height. We now have a height $z = 4$, associated with this point. Since $z = 4$, we need to rise 4 km above level from the location $(x, y) = (3, -2)$ in the xy plane. This will give us a uniquely defined point $(x, y, z) = (3, -2, 4)$.



GIVEN A LOCATION, FINDING A SET OF REPRESENTATIVE COORDINATES

Example Exercise 1.1.2: Find the coordinates (x, y, z), which can be used to describe location A.

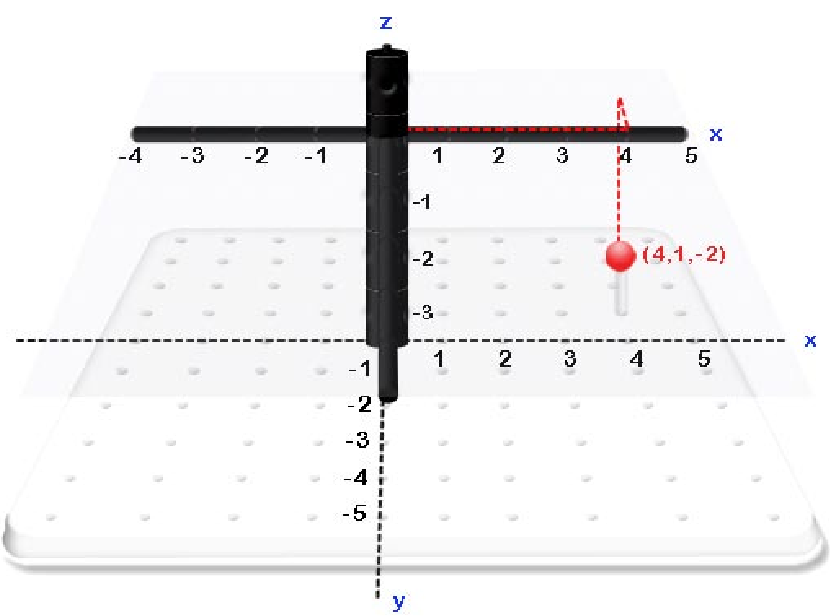
Solution: By starting in San Juan, traveling 3 km west, 4 km south, and 8 km up, we will arrive at location A. Hence the coordinates associated with A are $(x, y, z) = (-3, -4, 8)$.



Example Exercise 1.1.3: Find the location with coordinates (x, y, z) = (4, 1, -2).

Solution: We can start in San Juan at (x,y,z) = (0,0,0), and travel 4 km east of San Juan, then move 1 km north. Since z = -2, we need to travel 2 km below sea level, thus we go downward 2 spaces.

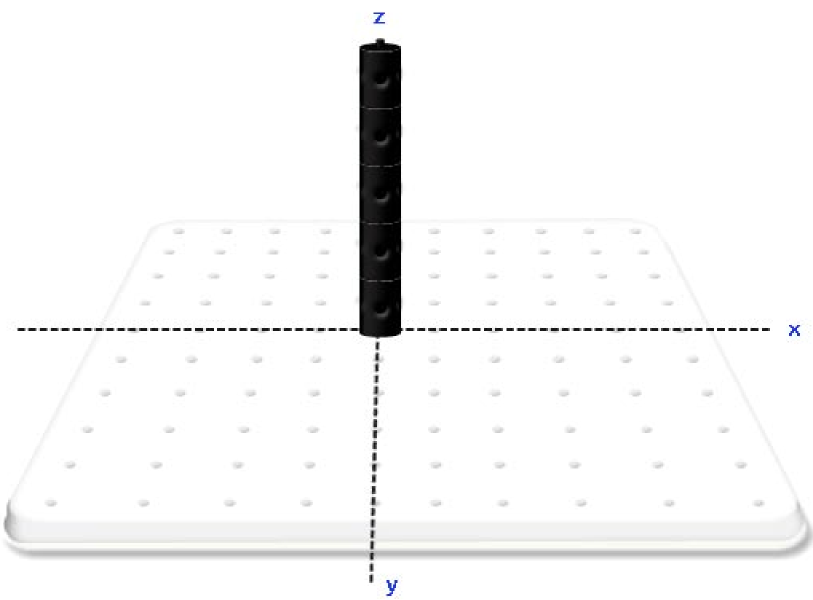
We place the point (4,1,-2) in the 3D kit as illustrated in the image below.



<u>**Generalization**</u>

Our coordinate system allows us to pass very simply between expressions of the form (x, y, z) = (a, b, c) and points in 3-space. In general, the coordinate system we will use in this textbook is defined as:

- x = units left/right of (0, 0, 0) (+ is right, - is left).
- y = units forward/backward of (0, 0, 0) (+ is forward, - is backward).
- z = units up/down of (0, 0, 0) (+ is up, - is down).



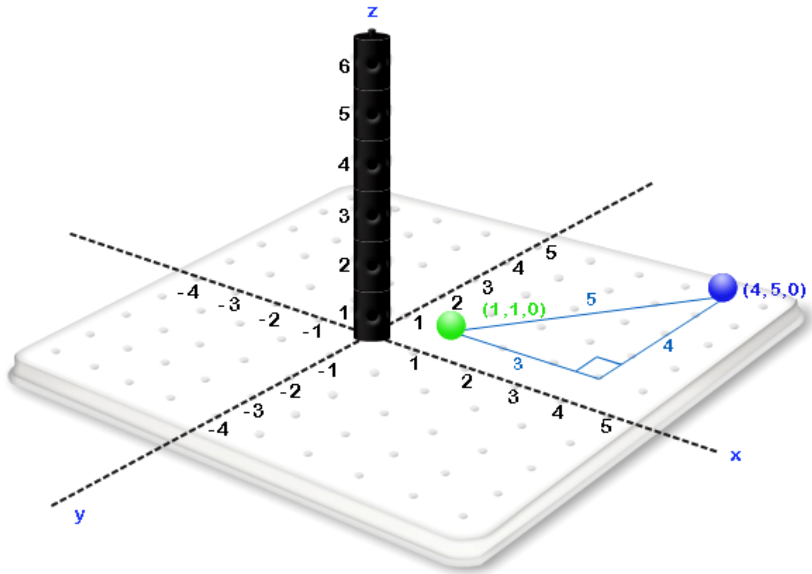
This is not the only way to define a system of coordinates in three dimensions. However, if we define the system in this way, x and y are the same as they have always been in the xy plane. We simply add the element of height, z, to each coordinate.

1.2 DISTANCE BETWEEN TWO POINTS IN 3-D

Example Exercise 1.2.1: Find the distance between the points (1, 1, 0) and (4, 5, 6).

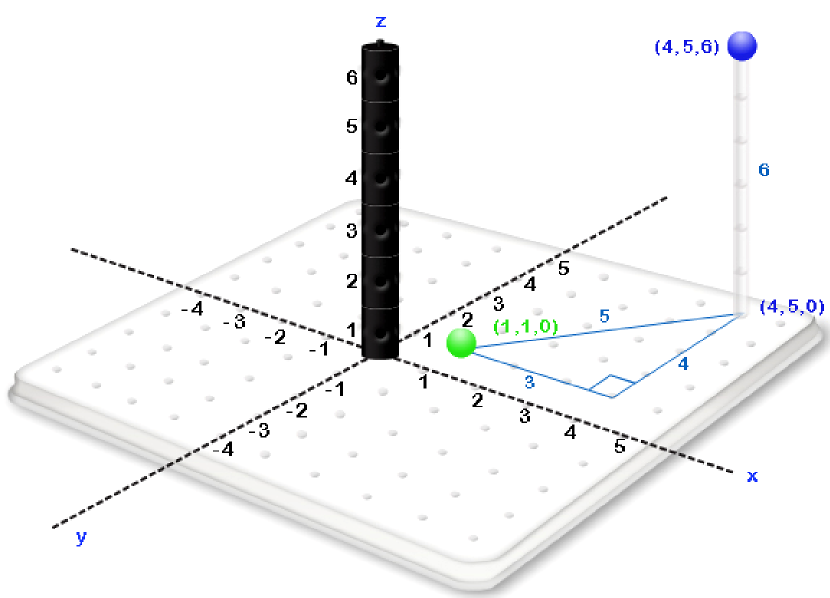
Solution:

1. Find the distance between the points (1, 1, 0) and (4, 5, 0) in the xy plane.



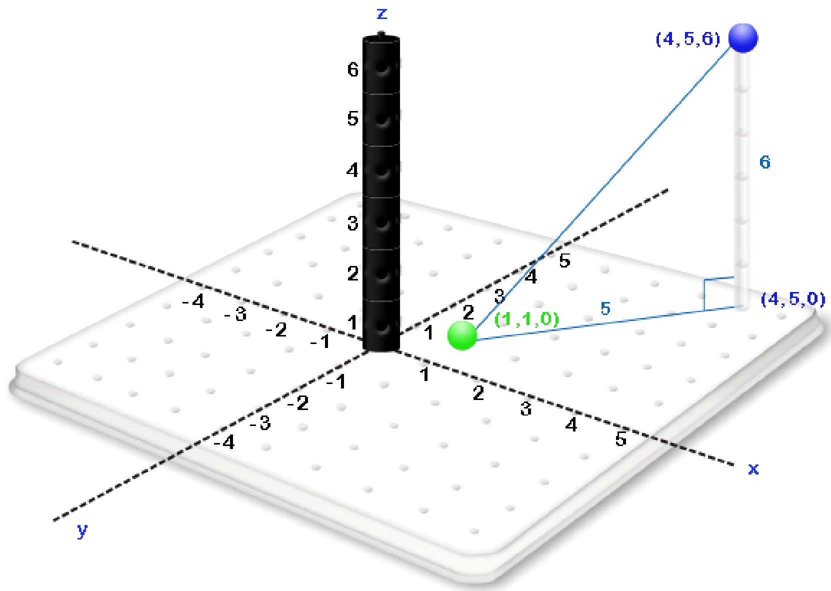
As this problem in two dimensions, the Pythagorean Theorem can be used to show that the distance between (1, 1, 0) and (4, 5, 0) is equal to 5.

2. Find the distance from (4, 5, 0) to (4, 5, 6).



The distance from (4, 5, 0) to (4, 5, 6) is clearly 6.

3. Form a right triangle with vertices (1, 1, 0), (4, 5, 0), and (4, 5, 6).



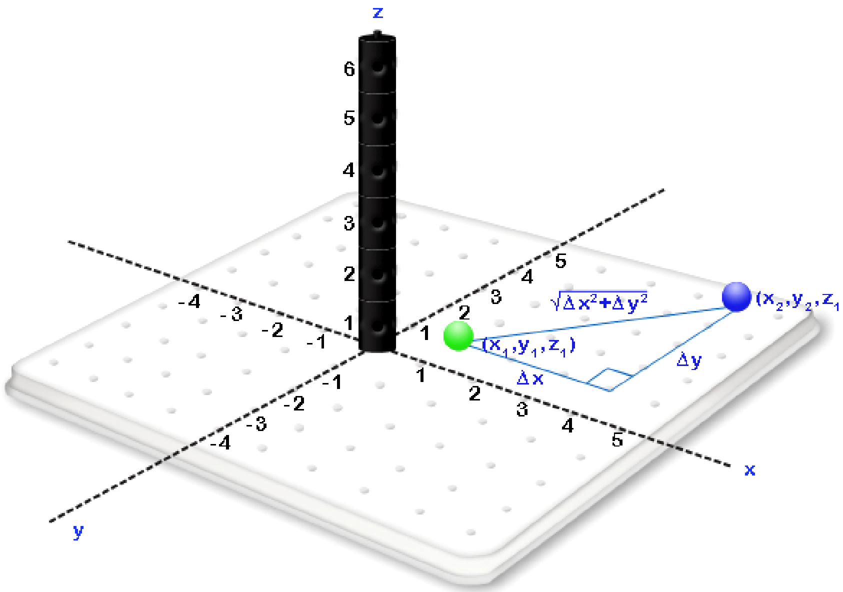
4. Use this right triangle and the Pythagorean Theorem to find the distance between (1, 1, 0) and (4, 5, 0):

   distance = $\sqrt{5^2+6^2}=\sqrt{61}$

Generalization

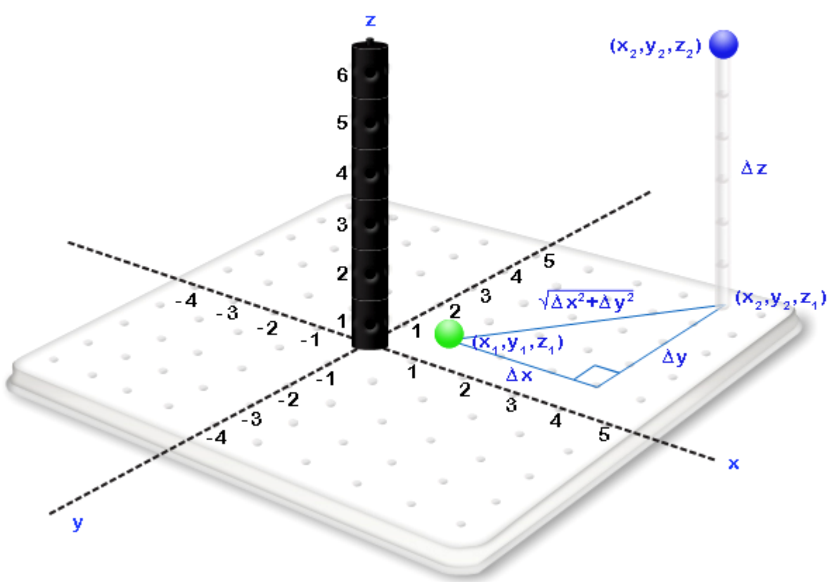
To find the distance between two arbitrary points (x_1, y_1, z_1) and (x_2, y_2, z_2), we complete the following steps:

1. Find the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) in the plane $z = z_1$.

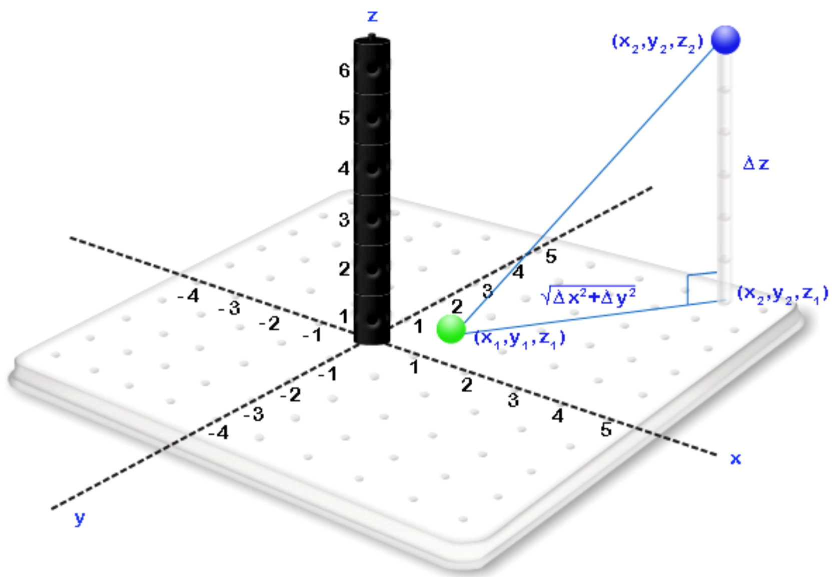


This is a problem in two dimensions so we can use the Pythagorean Theorem with $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ to show that the distance between (x_1, y_1, z_1) and (x_2, y_2, z_1) is $\sqrt{(\Delta x^2 + \Delta x^2)}$

2. Draw a line from (x_1, y_1, z_1) to (x_2, y_2, z_2) the distance of which will be Δz where $\Delta z = z_2 - z_1$.



3. Form a right triangle with vertices (x_1, y_1, z_1), (x_2, y_2, z_1), and (x_2, y_2, z_2).



4. With this right triangle, we can again use the Pythagorean Theorem to conclude that the distance from (x_1, y_1, z_1) to (x_2, y_2, z_2) is $\sqrt{\left(\sqrt{\Delta x^2 + \Delta y^2}\right)^2 + \Delta z^2} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. This is the general formula for the distance between two points in three dimensions.

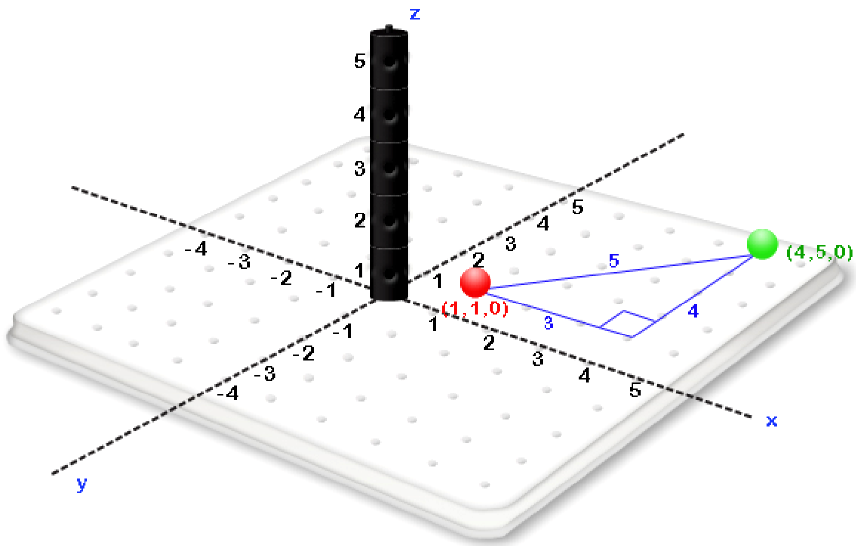
1.3 SLOPE OF A LINE SEGMENT BETWEEN TWO POINTS IN THREE DIMENSIONS

The geometric intuition for a slope is $\frac{rise}{run} =$ or our rise for every unit that we move.

Example Exercise 1.3.1: Find the slope of the segment from (1, 1, 0) to (4, 5, 6).

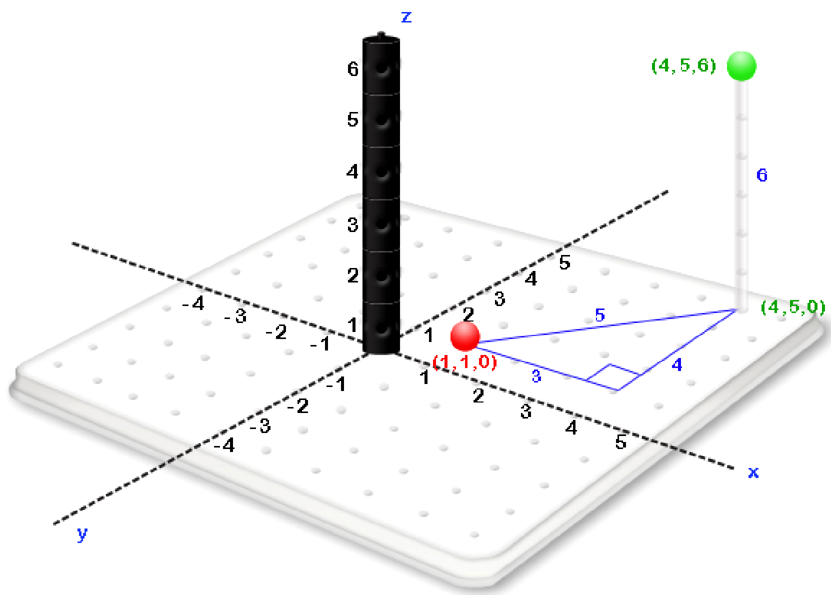
Solution:

1. Find the *run*: The *run* is between (1, 1, 0) and (4, 5, 0) in the *xy* plane.



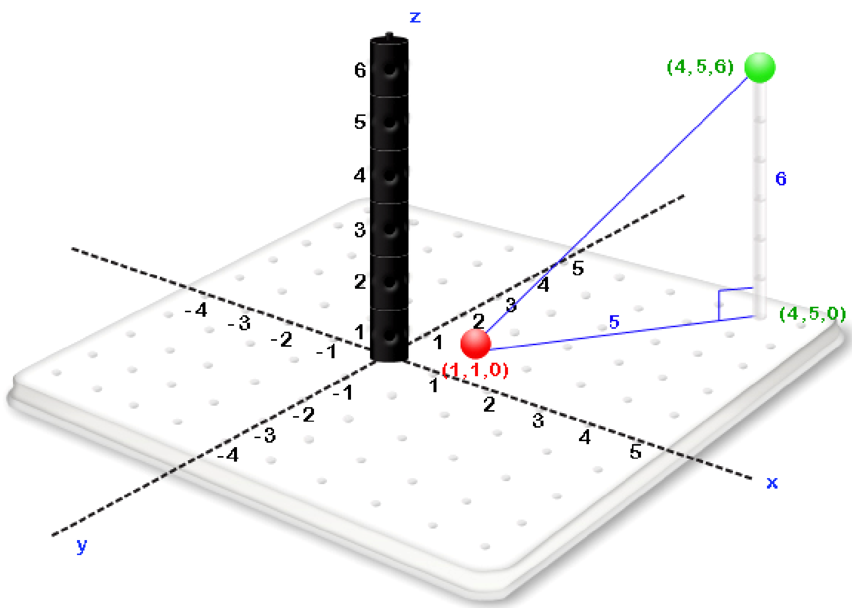
Using Pythagorean Theorem, we can conclude that the *run* is 5.

Find the *rise*:



The *rise* is the distance from (4, 5, 0) to (4, 5, 6) is 6.

To visualize the slope we are obtaining, place the right triangle with vertices (1, 1, 0), (4, 5, 0), and (4, 5, 6) in 3-space.

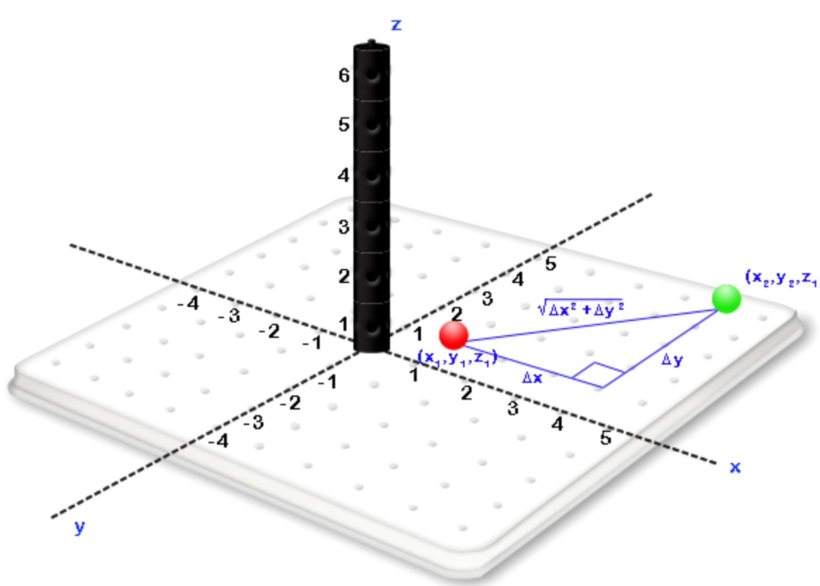


2. The slope is $\frac{rise}{run} = \frac{6}{5}$

Generalization

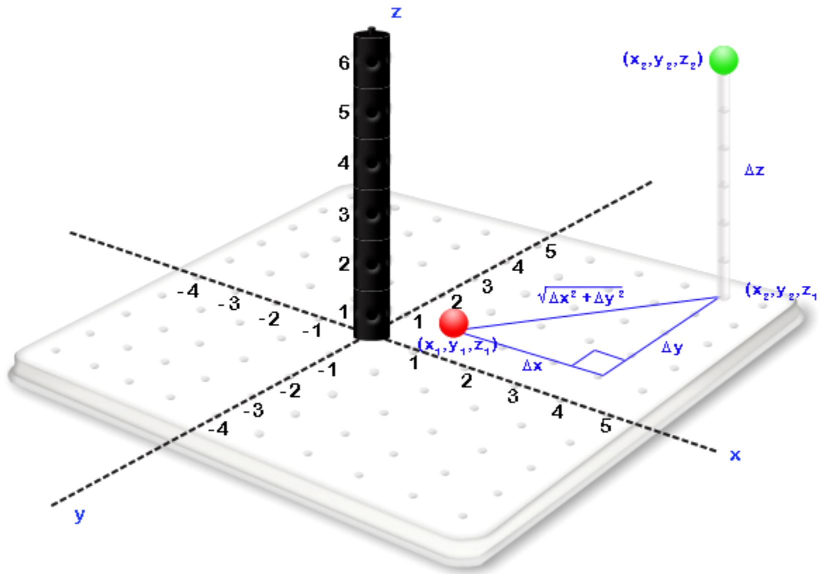
To find the slope of the segment that goes from one arbitrary point (x_1, y_1, z_1) to a second arbitrary point (x_2, y_2, z_1), we complete the following steps:

1. Find the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the plane $z = z_1$.

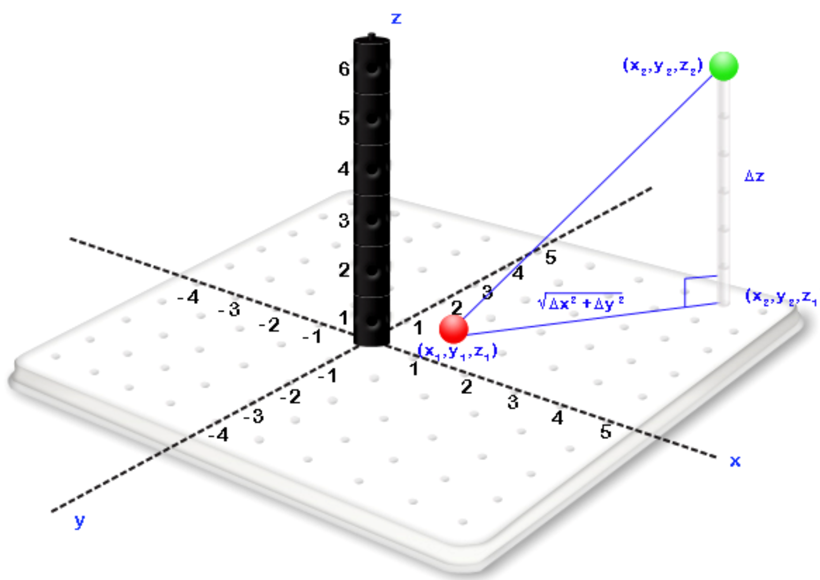


This is a problem in two dimensions so we can use Pythagorean Theorem with $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ to show that the distance between (x_1, y_1, z_1) and (x_2, y_2, z_1) is $\sqrt{\Delta x^2 + \Delta y^2}$

2. Draw a line from (x_1, y_1, z_1) to (x_2, y_2, z_2) the distance of which will be Δz where $\Delta z = z_2 - z_1$.



3. Form a right triangle with vertices (x_1, y_1, z_1), (x_2, y_2, z_1), and (x_2, y_2, z_2).



4. With this right triangle, we can conclude that the slope of the segment that goes from (x_1, y_1, z_1) to (x_2, y_2, z_2) is $\frac{rise_z}{run} = \frac{\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$. This is the general formula for the slope between two points in three dimensions. It is worth noting that the sign of the slope changes when we change our direction on the same line segment.

EXERCISE PROBLEMS:

1. Find 4 points (x,y,z) on your kit that satisfy the following equations (If there aren't four points that satisfy the equation, indicate that they do not exist):

a. $x = 3$
b. $y = 2$
c. $z = 5$
d. $x + y = 3$
e. $x + y = 3, z = 2$
f. $z = x + y$
g. $z = x + y, x = -1$
h. $z = x + y, z = 4$
i. $x = 3, y = -2$
j. $z = x^2 + y^2$
k. $z = x^2 + y^2, x = -1$
l. $z = x^2 + y^2, z = 4$
m. $z = x^2 + y^2, z = -1$

2. For the following sets of points ***A*** and ***B***, do the following:

i. Construct a right triangle on your kit where the two sides that are not the hypotenuse are horizontal and vertical.
ii. Find the distance from ***A*** to ***B***
iii. Find the slope from ***A*** to ***B***
iv. Find the slope from ***B*** to ***A***

A. ***A*** = (1,2,5) and ***B*** = (4,2,5)
B. ***A*** = (3,2,1) and ***B*** = (3,6,5)
C. ***A*** = (4,4,3) and ***B*** = (0,1,1)
D. ***A*** = (1,-4,2) and ***B*** = (-2,6,-1)
E. ***A*** = (7,2,-3) and ***B*** = (4,-2,3)
F. ***A*** = (3,0,-1) and ***B*** = (3,-4,4)
G. ***A*** = (-4,-4,4) and ***B*** = (-5,3,-2)

H. $\boldsymbol{A}$ = (-1,-2,-3) and $\boldsymbol{B}$ = (-1,-1,2)

PRACTICE PROBLEMS

PP 1.1.1

Place the points (0, -1, 2), (-2, -3, 0), (3, 0, 4), and (-4, 5, 3) in your 3D Kit.

PP 1.1.2

a. Label your 3D Kit so that x goes from $x = -4$ to $x = 5$, y goes from $y = -4$ to $y = 5$, and z goes from $z = 0$ to $z = 5$.

b. Place the point associated with the coordinates $(x, y, z) = (2, 1, 2)$.

c. Place the point associated with the coordinates $(x, y, z) = (0, 3, 4)$.

d. Place the point directly between these two points. What are the coordinates of this point?

e. In general, what do you think will be a formula for the point directly between two points (x_0, y_0, z_0) and (x_1, y_1, z_1)?

PP 1.1.3

a. Label your 3D Kit so that x goes from $x = -4$ to $x = 5$, y goes from $y = -4$ to $y = 5$, and z goes from $z = 0$ to
$z = 5$ where the units are in kilometers.

b. Place the point associated with the coordinates $(x, y, z) = (4, -3, 2)$ on your 3D Kit.

c. Starting at the point $(x, y, z) = (4, -3, 2)$, move 7 km units west, 2 km north, and 1 km down and use the poles and spheres to place the resulting point. What are the coordinates of this point?

PP 1.1.4

a. Obtain four points that satisfy the equation $y = 1$.

b. Place the four points in the 3D Kit coordinate system.

c. Place the plane that you believe will satisfy that $y = 1$ in the 3D Kit.

d. Find 3 points on the plane you've placed in our coordinate system that are different from the initial four points that you placed and identify the coordinates of each of them.

e. Verify that each of the points in part (d) satisfies the equation $y = 1$.

PP 1.1.5

a. Represent the line that passes through the points: a) (-1,3,3), (4,3,5), and b) (-1,-4,3), (5,-4,1) in your 3D Kit. Use the 3D Kit to represent another line that is in the *x* direction.

b. Represent the line that passes through the points: a) (-2,-3,2), (-2,3,6), and b) (4,-3,6), (4,5,2) in your 3D Kit. Use the 3D Kit to represent another line that is in the *y* direction.

PP 1.1.6

a. Place the points (2, 3, 5), (-4, 3, 2), and (2, -4, 3) in your 3D Kit.

b. By just looking at the points on the 3D Kit decide which two of the points are in the *x* direction and which two are in the *y* direction.

PP 1.2.1

a. On the base of your 3D Kit, use Pythagoras to find the distance between the points (2, 3, 0) and (4, 1, 0).

b. Find and represent the distance from (4, 1, 0) to (4, 1, 3) on your 3D Kit.

c. Construct the triangle with vertices (2, 3, 0), (4, 1, 0), and (4, 1, 3). How do we know that this is a right triangle?

d. Indicate the lengths of the two known sides of this right triangle and use Pythagoras to obtain the remaining side.

e. Use the results of parts (a), (b), (c), and (d) to obtain the distance between the points (2, 3, 0) and (4, 1, 3).

PP 1.2.2

a. Consider the base of your 3D Kit to be the plane $z = 2$. Indicate the points (-2, 3, 2) and (-3, -4, 2). Use Pythagoras to find the distance between them.

b. Place a pole that goes from (-3, -4, 2) to (-3, -4, 4). What is the length of this pole?

c. Construct the triangle with vertices (-2, 3, 2), (-3, -4, 2), and (-3, -4, 4). How do we know that this is a right triangle?

d. Indicate the lengths of the two known sides of this right triangle and use Pythagoras to obtain the remaining side.

e. Use the results of parts (a), (b), (c), and (d) to obtain the distance between the points (-2, 3, 2) and (-3, -4, 4).

PP 1.3.1

Use your 3D Kit to represent the points *Q*(4, 1, 5) and *P*(1, 1, 2). Use these points together with (4, 1, 2) to form a right triangle. Use the right triangle to represent and compute the rise and the run. Use the 3D Kit to explain why the slope is positive. Represent a segment that will have a negative slope in your 3D Kit.

PP 1.3.2

a. Represent the points *P*(3, -4, 2) and *Q*(-2, -4, 4) on the 3D Kit.

b. Use entirely geometric arguments to find the slope of the line through *P* and *Q*.

PP 1.3.3

a. Represent the points *P* (3, -4, 2) and *Q* (3, 3, 5) on the 3D Kit.

b. Use entirely geometric arguments to find the slope of the line through *P* and *Q*.

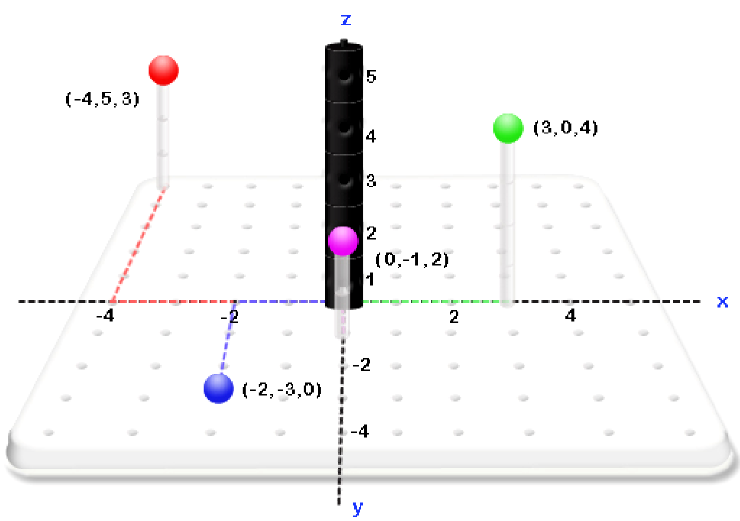
PP 1.3.4

Follow the following steps to find the slope from (2,3,0) to (4,1,3)

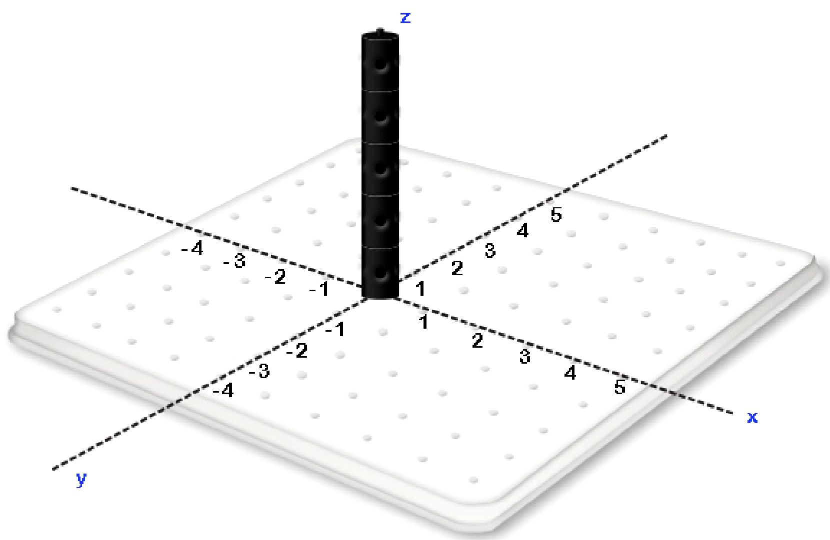
a. Place the right triangle that contains the rise and run that are needed to find the slope from (2,3,0) to (4,1,3)
b. On the base of your 3D Kit, use Pythagoras to find the run associated with this slope. I.e., the distance between the points (2, 3, 0) and (4, 1, 0).
c. Find the distance associated with the rise of this triangle on your 3D kit. I.e., the distance from (4, 1, 0) to (4, 1, 3).
d. Use the results of parts (a), (b), and (c) to obtain the slope of the segment that goes from the point (2, 3, 0) to the point (4, 1, 3).

SOLUTIONS:

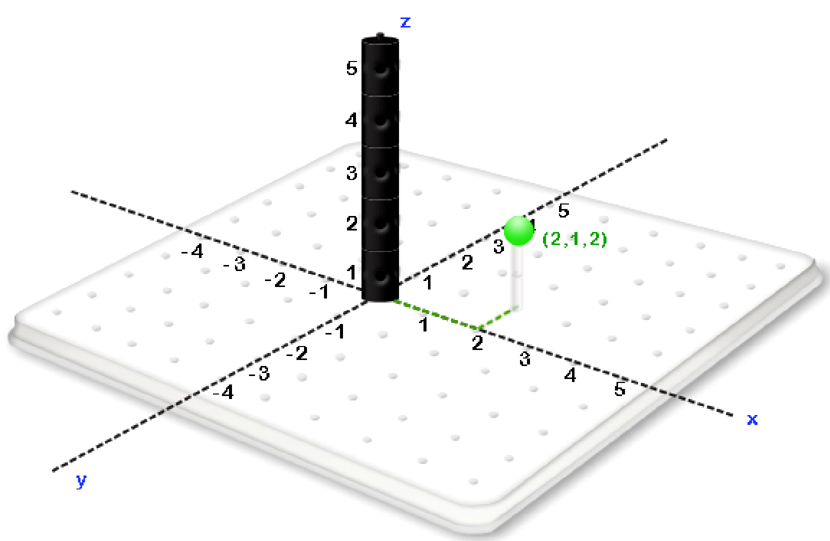
PP 1.1.1



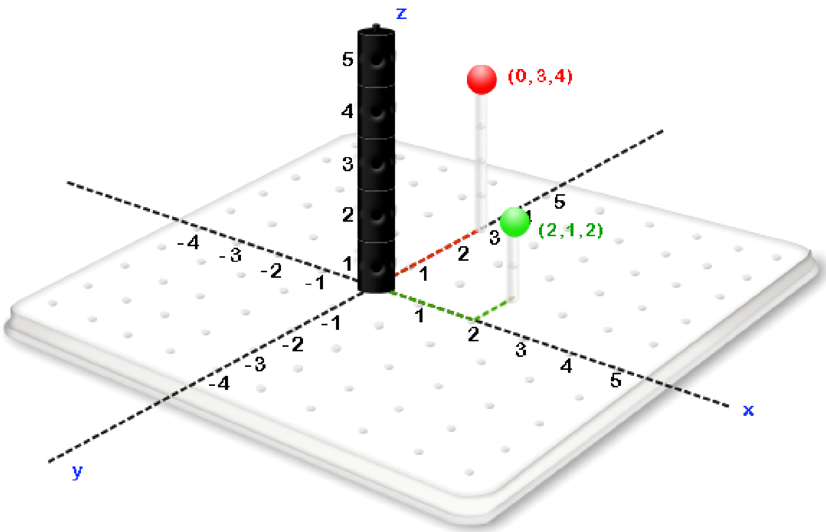
PP 1.1.2



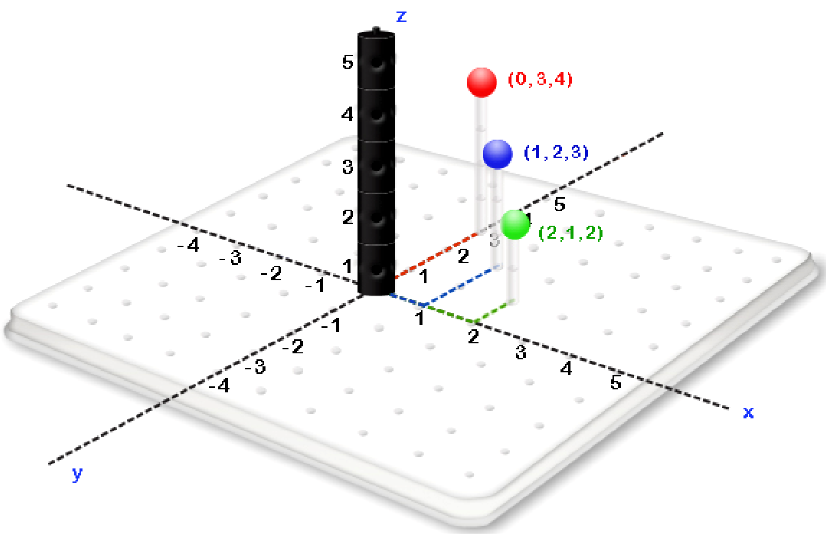
a.



b.



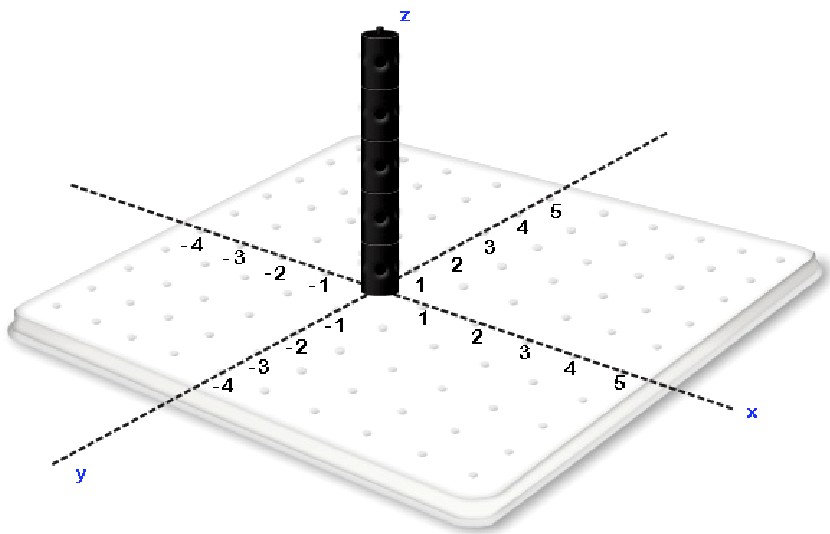
c.



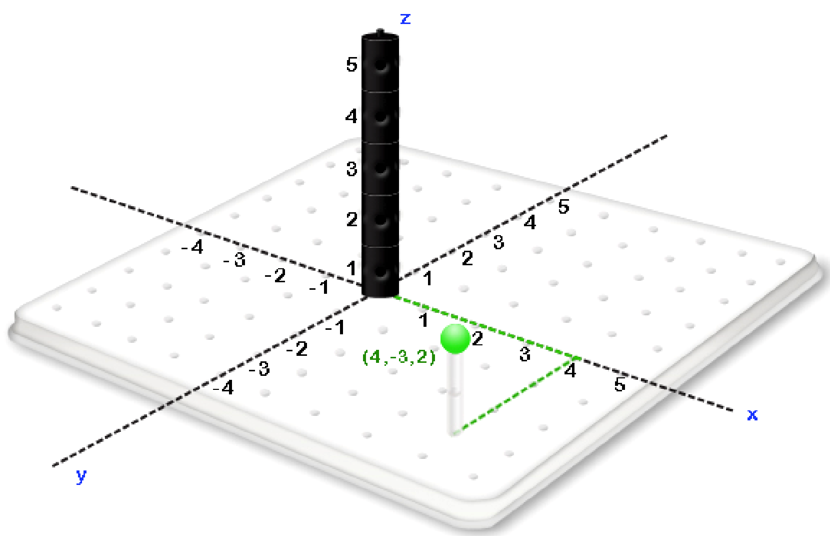
d.

e. $\left(\dfrac{x_0+x_1}{2},\dfrac{y_0+y_1}{2},\dfrac{z_0+z_1}{2}\right)$

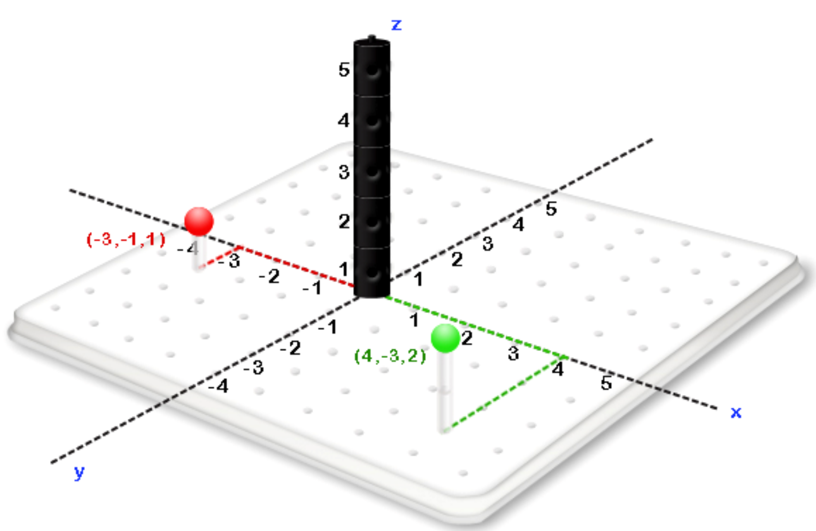
PP 1.1.3



a.



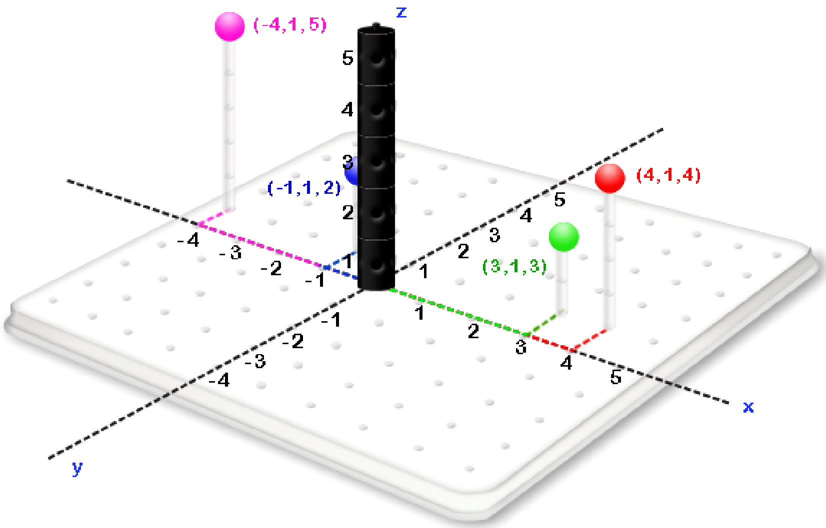
b.



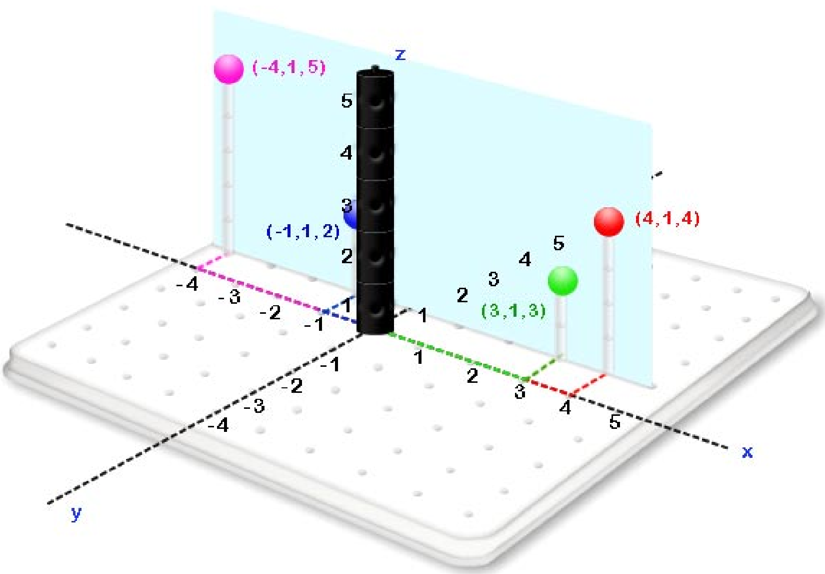
c.

PP 1.1.4

a. (-4, 1, 5), (-1, 1, 2), (3, 1, 3) and (4, 1, 4).

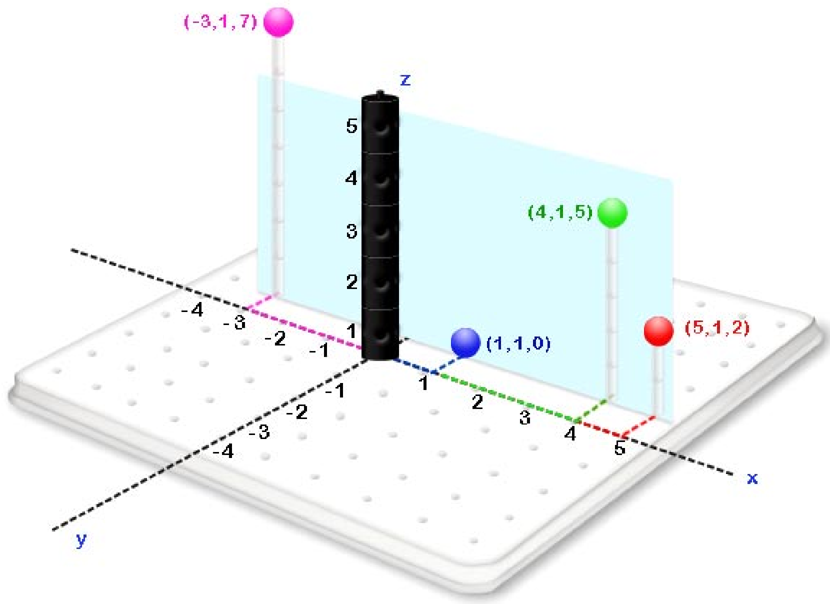


b.



c.

d. Some of the possible points are:



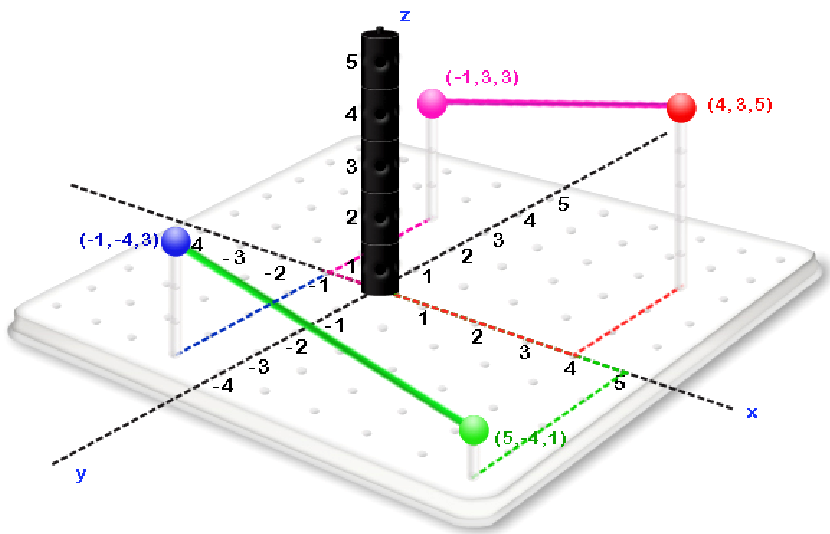
e. (-3, 1, 7) = (x, y, z)
y = 1

(4, 1, 5) = (x, y, z)
y = 1

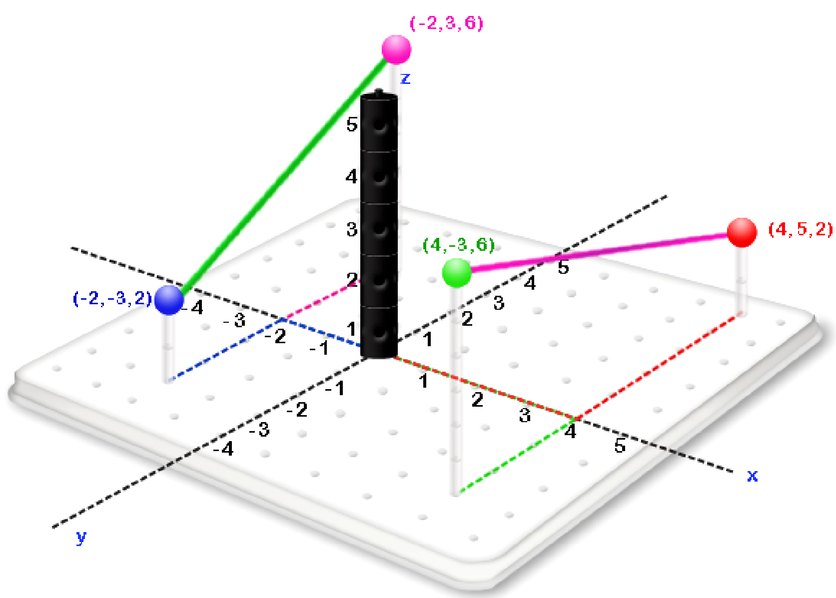
(5, 1, 2) = (x, y, z)
y = 1

(1, 1, 0) = (x, y, z)
y = 1

PP 1.1.5

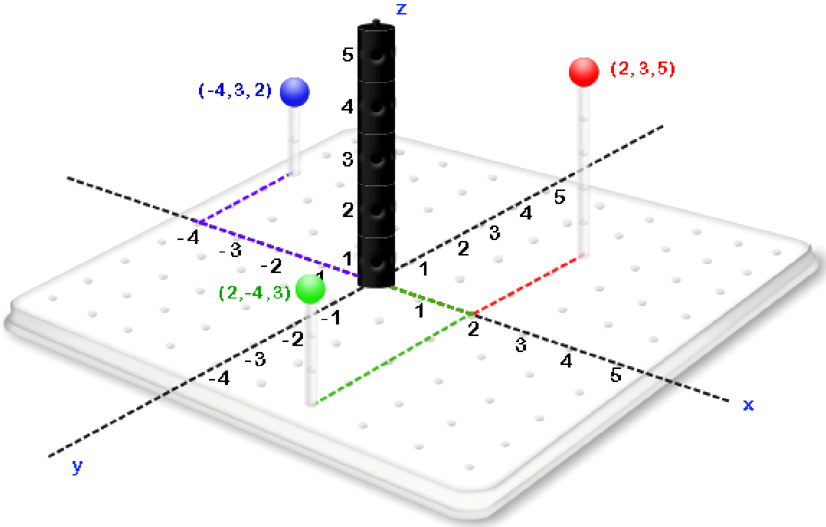


a.



b.

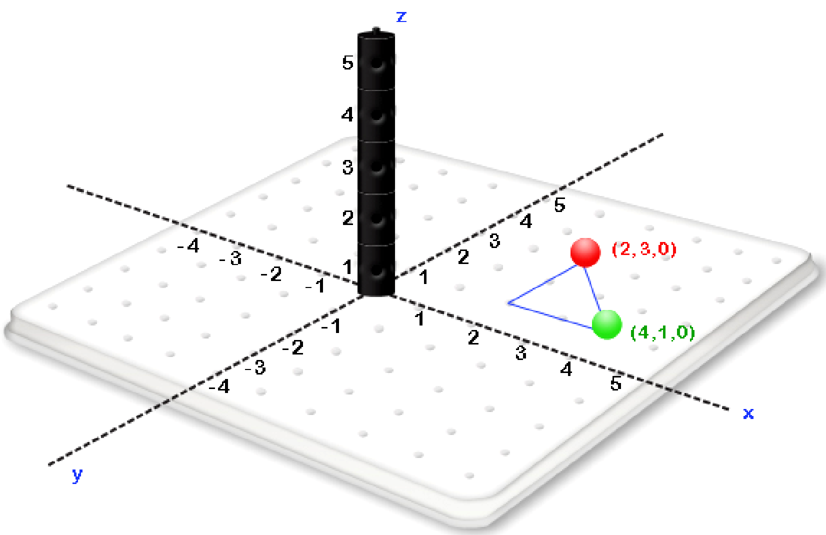
PP 1.1.6



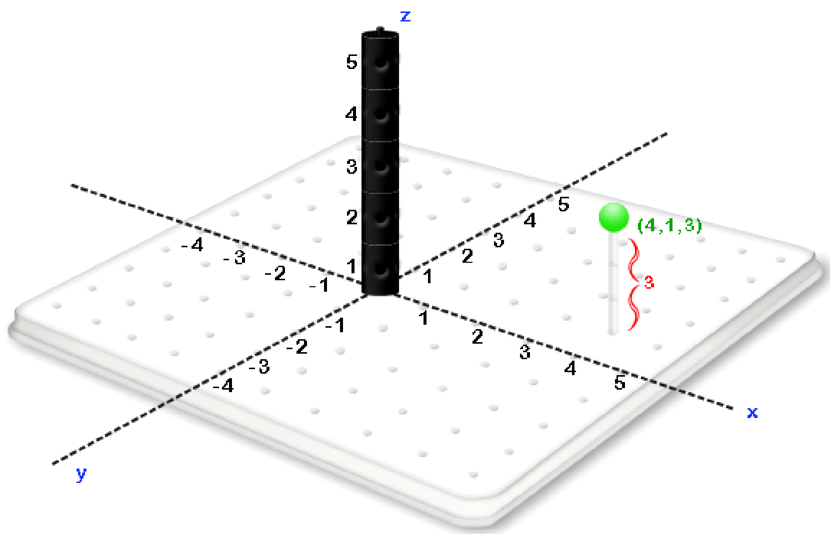
a.

b. The points in the *y* direction are: (2, 3, 5) and (2, -4, 3). The points in the *x* direction are: (2, 3, 5) and (-4, 3, 2).

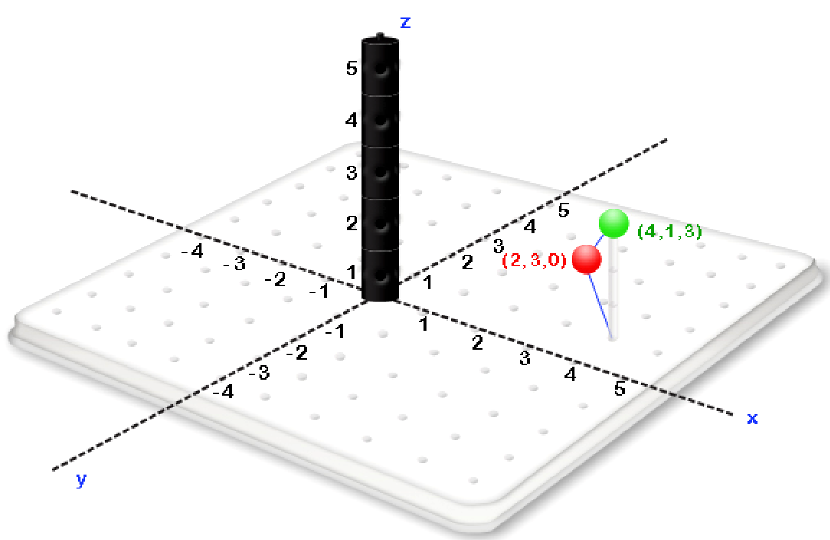
PP 1.2.1



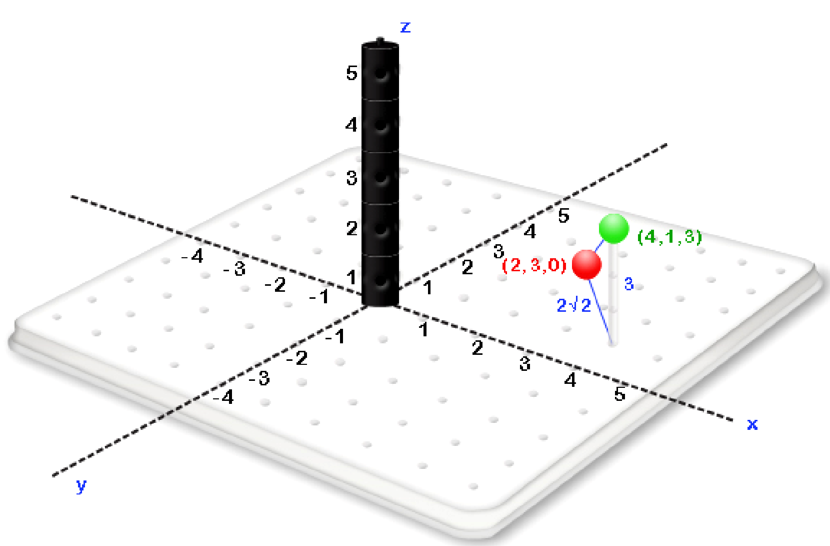
a. $c = \sqrt{(4-2)^2 + (1-3)^2} = \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = 2\sqrt{2}$


z
5
4
3
2
1
-4
-3
-2
-1
1
2
3
4
5
(4,1,3)
3
-1
-2
-3
-4
1
2
3
4
5
x
y

b.


z
5
4
3
2
1
-4
-3
-2
-1
1
2
3
4
5
(4,1,3)
(2,3,0)
-1
-2
-3
-4
1
2
3
4
5
x
y

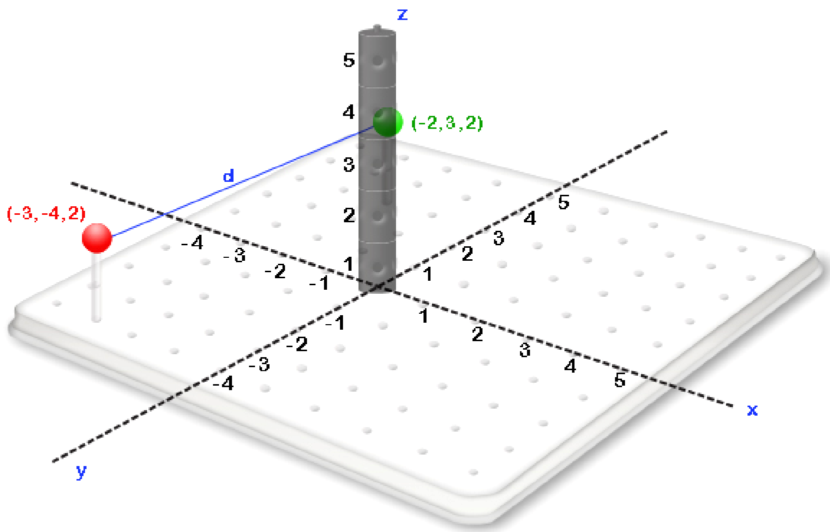
c.


z
5
4
3
2
1
-4
-3
-2
-1
1
2
3
4
5
(4,1,3)
(2,3,0)
3
2√2
-1
-2
-3
-4
1
2
3
4
5
x
y

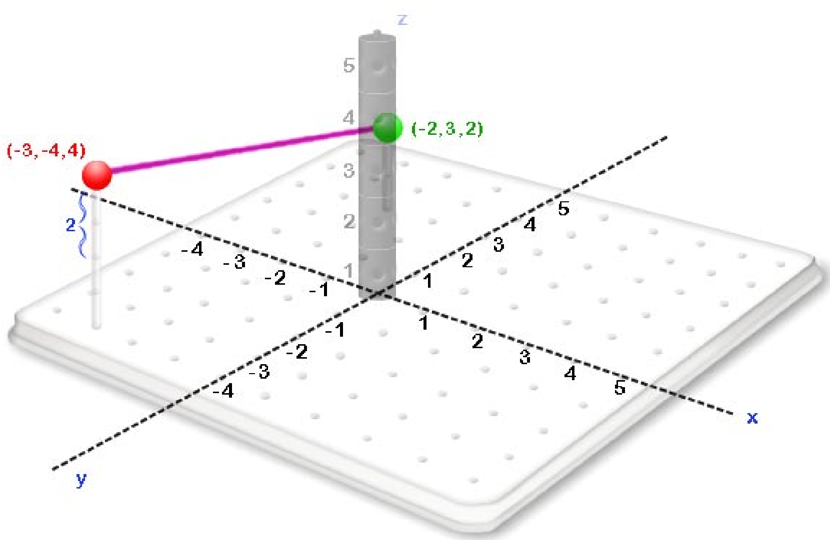
d.

e. $c = \sqrt{\left(2\sqrt{2}\right)^2 + 3^2} = \sqrt{8+9} = \sqrt{17}$

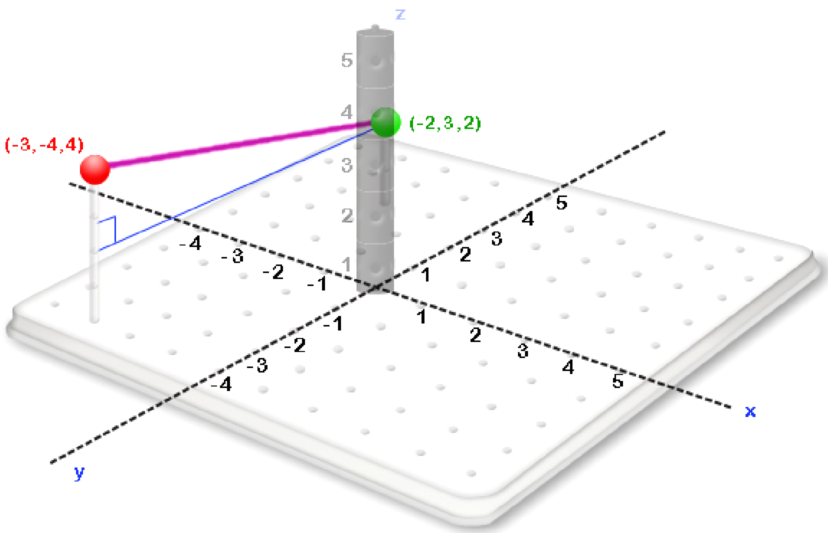
PP 1.2.2



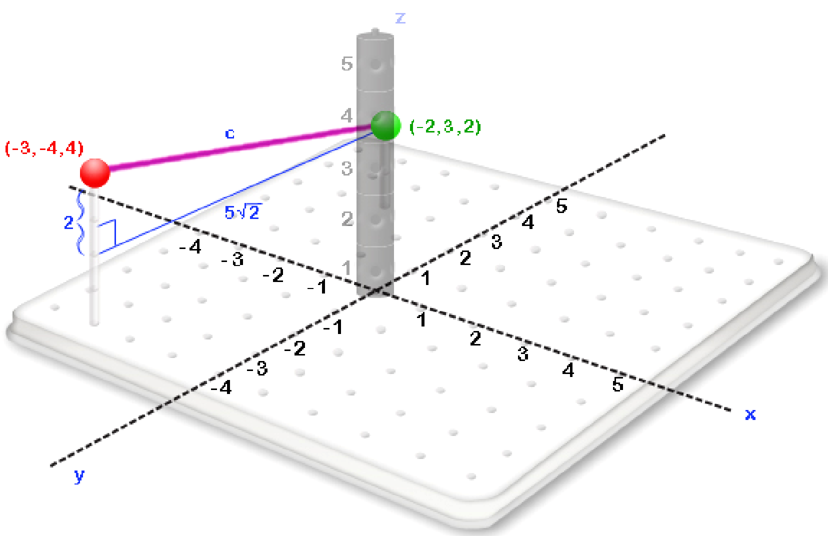
a. $d = \sqrt{\left(1^2 + 7^2\right)} = \sqrt{1+49} = \sqrt{50} = \sqrt{(25)(2)} = 5\sqrt{2}$



b.



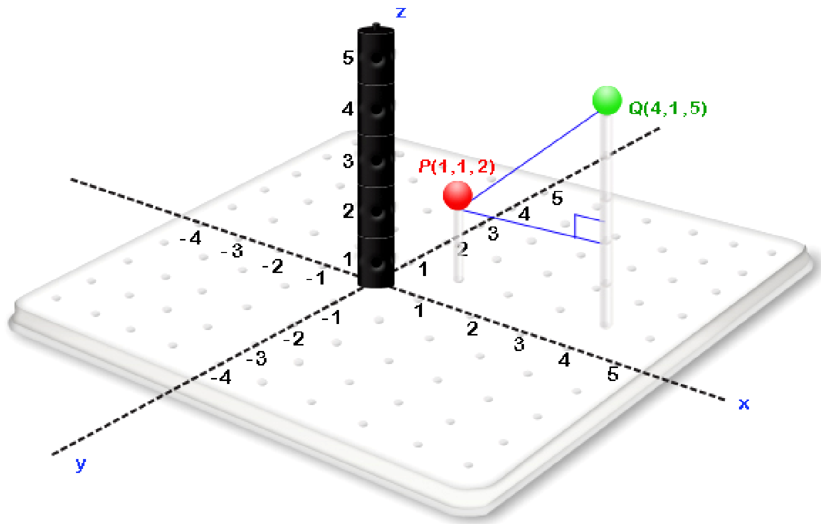
c.



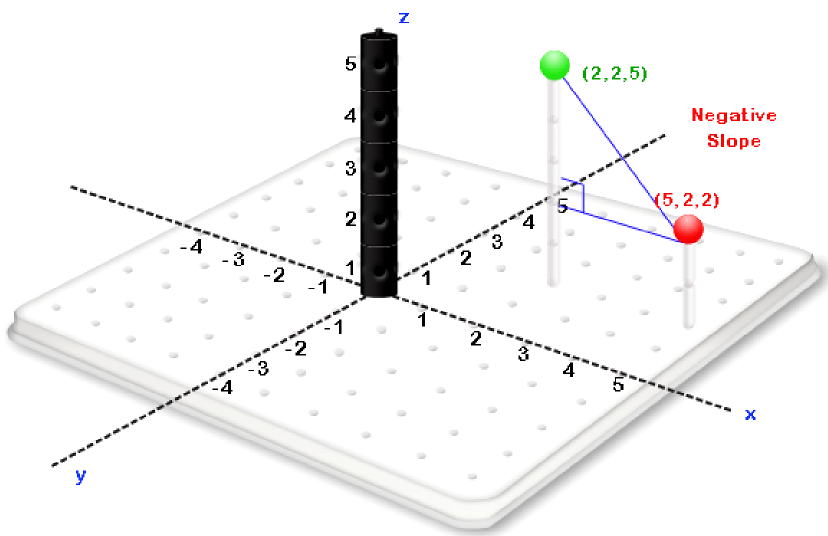
d.

e. $c = \sqrt{\left(5\sqrt{2}\right)^2 + 2^2} = \sqrt{4+50} = \sqrt{54} \qquad = \sqrt{(9)(6)} = 3\sqrt{6}$

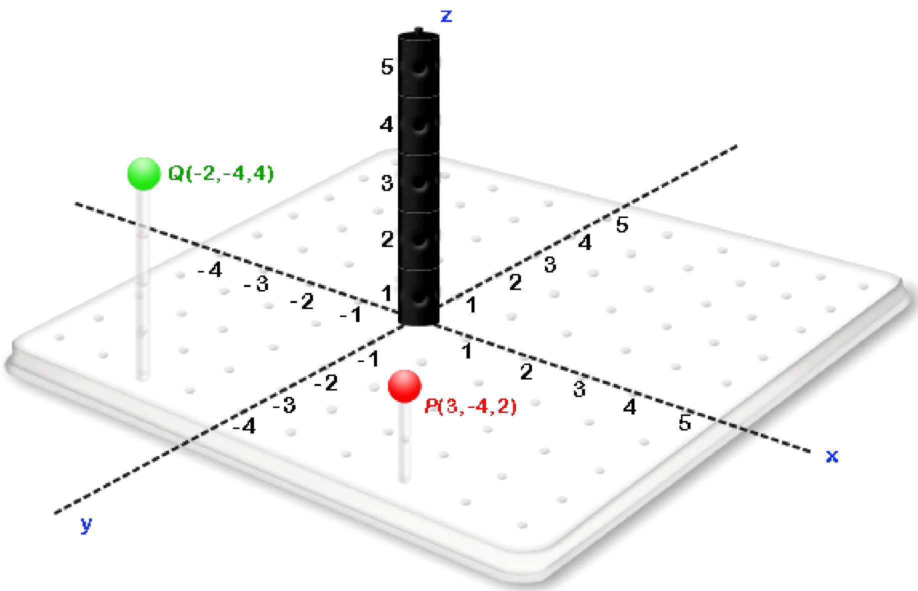
PP 1.3.1



The slope is positive because as *x* increases the height *z* also increases.

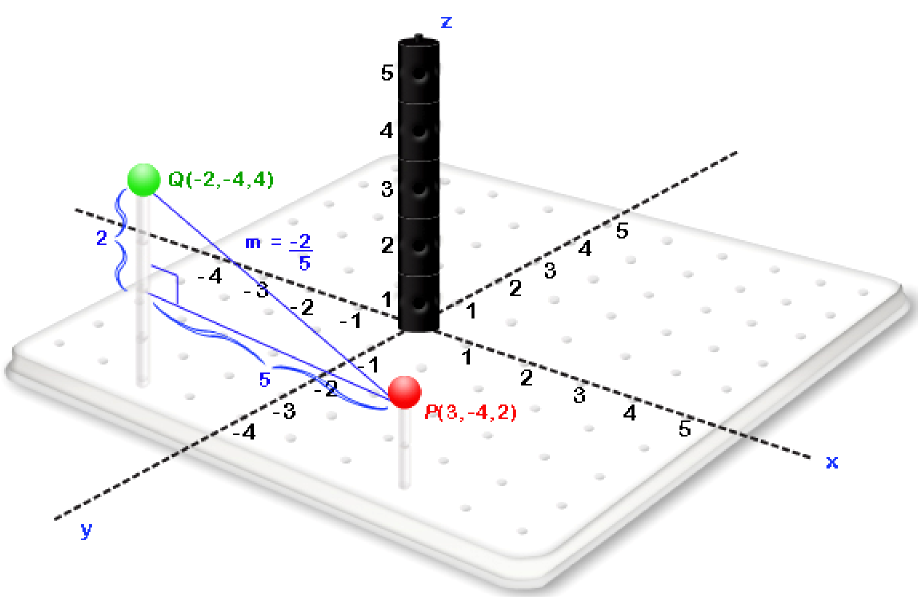


PP 1.3.2

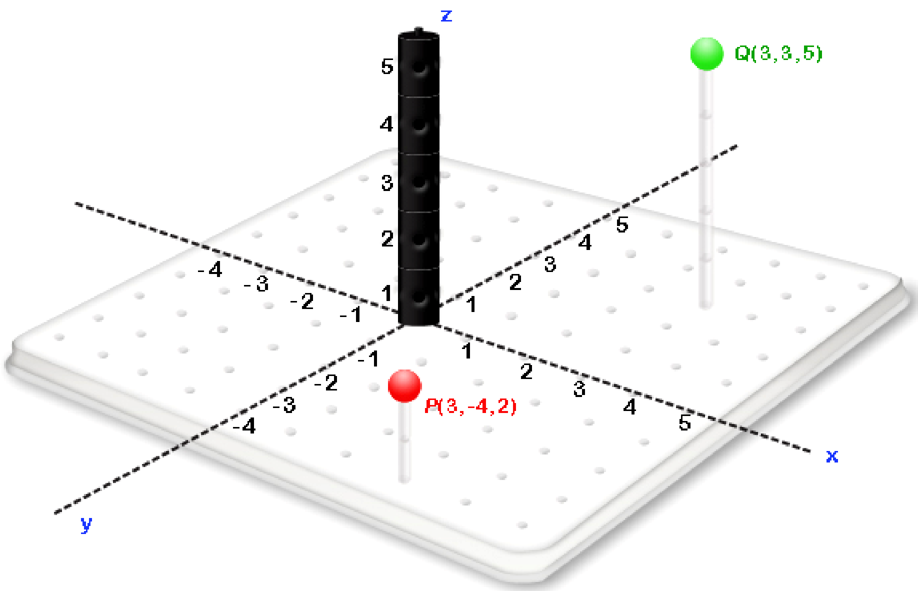


a.

b. The slope is negative because as *x* increases the height *z* also decreases.

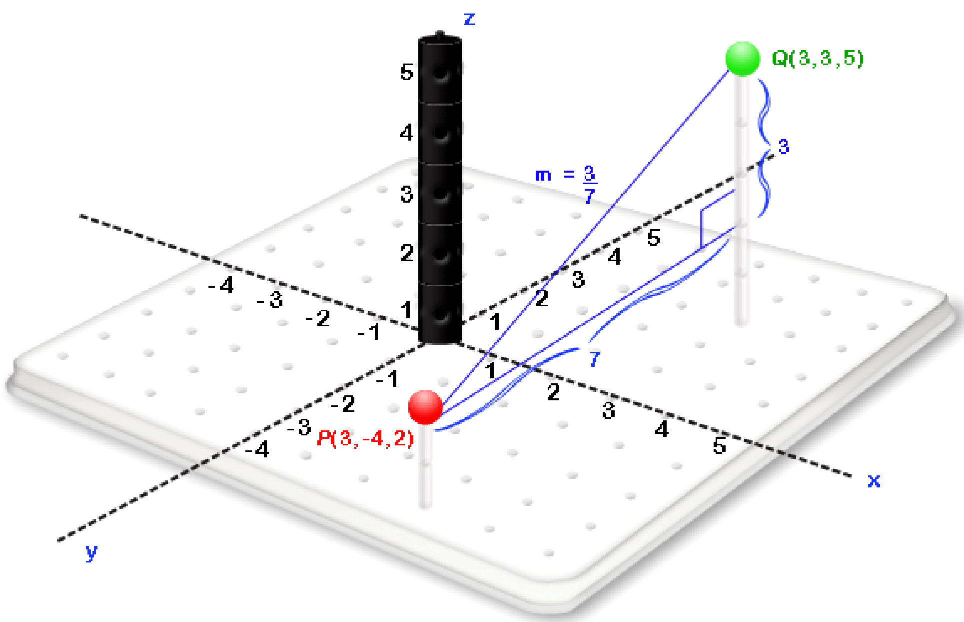


PP 1.3.3

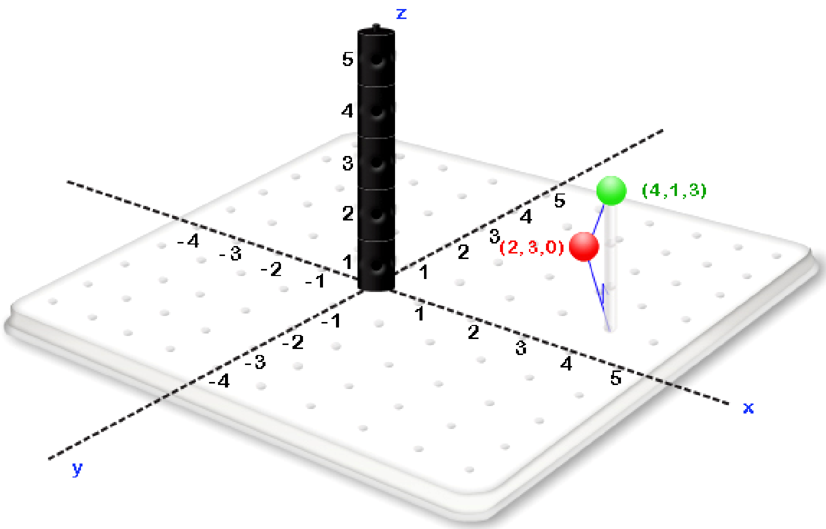


a.

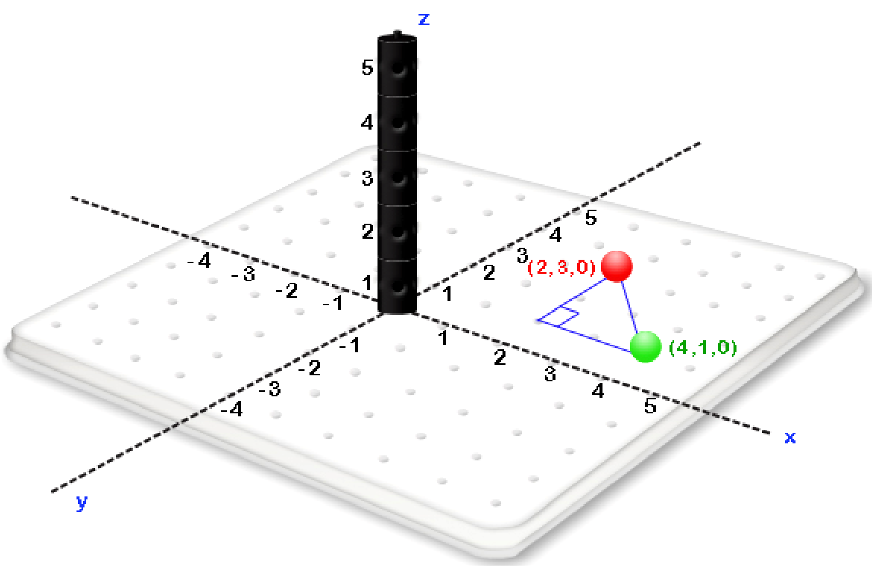
b. The slope is positive because as *y* increases the height *z* also increases.



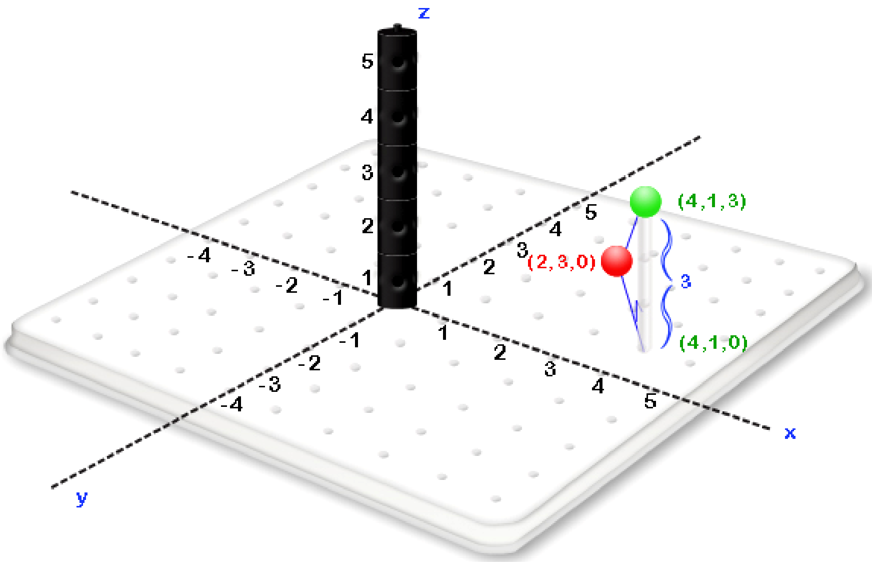
PP 1.3.4



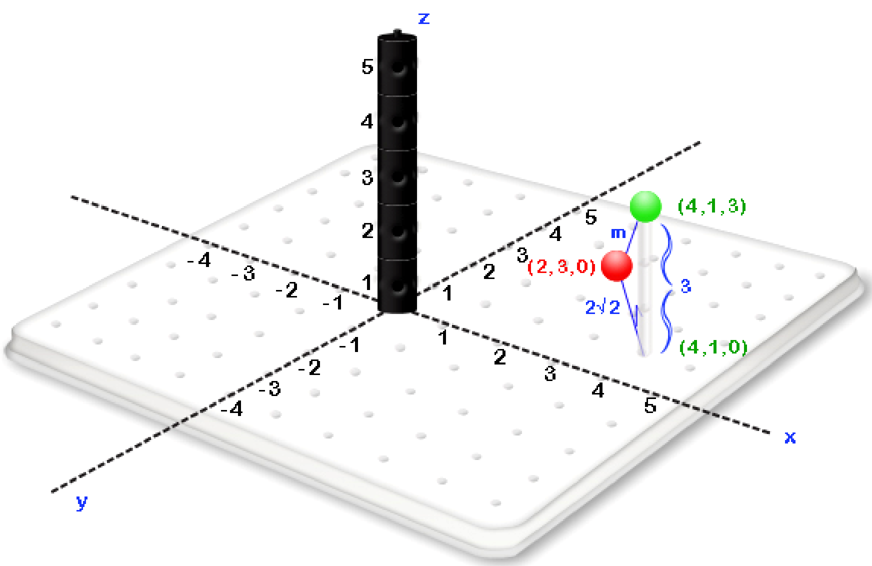
a. The above right triangle contains the necessary *rise* and *run* to find the slope from (2,3,0) to (4,1,3) **.**



b. ***run*** = $\sqrt{2^2+2^2} = 2\sqrt{2}$



c. ***rise*** = 3



d. $\mathbf{m} = \dfrac{rise}{run} = \dfrac{3}{2\sqrt{2}}$